

DESY 96 –165
ITP–UH–15/96



hep-th/9608131
revised version

(4,4) SUPERFIELD SUPERGRAVITY ¹

Sergei V. Ketov, ² and Christine Unkmeir

*Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
ketov@itp.uni-hannover.de*

and

Sven-Olaf Moch

*Deutsches Elektronen Synchrotron, DESY
Notkestraße 85, 22603 Hamburg, Germany
moch@mail.desy.de*

Abstract

We present the N=4 superspace constraints for the two-dimensional (2d) off-shell (4,4) supergravity with the superfield strengths expressed in terms of a (4,4) twisted (scalar) multiplet TM-I, as well as the corresponding component results, in a form suitable for applications. The constraints are shown to be invariant under the N=4 super-Weyl transformations, whose N=4 superfield parameters form another twisted (scalar) multiplet TM-II. To solve the constraints, we propose the Ansatz which makes the N=4 superconformal flatness of the N=4 supergravity curved superspace manifest. The locally (4,4) supersymmetric TM-I matter couplings, with the potential terms resulting from spontaneous supersymmetry breaking, are constructed. We also find the full (4,4) superconformally invariant (improved) TM-II matter action. The latter can be extended to the (4,4) locally supersymmetric Liouville action which is suitable for describing (4,4) supersymmetric non-critical strings.

¹Supported in part by the ‘Deutsche Forschungsgemeinschaft’ and the NATO grant CRG 930789

² On leave of absence from: High Current Electronics Institute of the Russian Academy of Sciences,

Siberian Branch, Akademichesky 4, Tomsk 634055, Russia

1 Introduction

A full off-shell structure of any supersymmetric field theory most naturally exhibits itself in superspace, provided the superfield formulation of the theory in terms of unconstrained superfields (the so-called *prepotentials*) is available. This is particularly relevant for the supergravity theories, which are usually formulated in superspace by using the Wess-Zumino-type constraints [1] (see refs. [2, 3, 4] for a review.) A fully covariant superfield formulation is desirable for quantisation purposes, as well as for renormalisation or a finiteness check. A covariant superspace solution is also useful for studies of super-Riemannian surfaces and the associated super-Beltrami differentials, where conformal gauge may not be convenient and light-cone gauge may not be accessible, e.g. as far as the higher-genus string and superstring amplitudes are concerned [5].

Once a full set of auxiliary fields needed to close the supersymmetry algebra in a supersymmetric field theory is known, it should be possible to solve the equivalent superspace constraints. In four dimensions, the full solution to the N=1 superspace supergravity is known for a long time [6], whereas solving the N=2 extended superspace supergravity presumably requires the use of the N=2 harmonic superspace [7], with the necessarily infinite number of auxiliary fields. As far as the four-dimensional N=2 supergravity in the ordinary N=2 superspace is concerned, only linearised solutions were found so far [8, 9].

In *two dimensions* (2d), where the Lorentz group is more restricted, it should be possible to find full covariant solutions to the (p, q) -extended supergravities in the ordinary (p, q) -extended superspace, whenever the corresponding off-shell formulation is available, i.e. if $p, q \leq 4$. Indeed, the fully covariant solutions are already known for $(1, 0)$ [10], $(1, 1)$ [11], $(p, 0)$ [12] and $(2, 2)$ [13] supergravities. In particular, the solution to the 2d, $(2, 2)$ supergravity can also be obtained by dimensional reduction from *four dimensions* (4d). Though being not practical for solving superspace constraints, the method of dimensional reduction is nevertheless useful for getting insights into the complicated component structure of extended supergravities, and for spontaneous supersymmetry breaking as well (see sect. 4 for an example).

To the best of our knowledge, no attempts were ever made towards solving the covariant 2d off-shell (4,4) superspace supergravity constraints, since they were first formulated by Gates *et. al.* in ref. [14] (see also the related work [15]). Recently, Grisaru and Wehlau [16, 17] found the complete covariant solution to the 2d, $(2, 2)$ supergravity constraints in the ordinary N=2 superspace, as well as the corresponding superspace measures and invariant actions. It was achieved, in part, by working in a

proper light-cone-type basis, rather than using the gamma matrices as in refs. [14, 15]. In this paper, we begin the similar program for the case of the 2d, (4,4) superspace supergravity. Surprisingly enough, as far as the solution to the (4,4) supergravity constraints is concerned, it turns out to be possible to follow the lines of the N=2 solution up to a Wess-Zumino-type supersymmetric gauge fixing. The gauge-fixing should result in only one irreducible (4,4) superfield describing the off-shell N=4 conformal supergravity multiplet. It is related to the fact that the general (4,4) vector superfield H^m has many redundant supersymmetric gauge degrees of freedom, unlike its N=2 counterpart. The relevant irreducible superfield can be rather easily identified in the linearised approximation [18]. Gauging away the rest of the N=4 superfields does not introduce propagating ghosts, despite of a high degree of non-linearity. The supersymmetric gauge-fixing in the (4,4) superspace supergravity is however beyond the scope of this paper.

We also present here some interesting new features for 2d couplings of the twisted chiral matter multiplets, TM-I and TM-II, to the (4,4) supergravity. In particular, we show how to generate the potential terms via spontaneous N=4 supersymmetry breaking by dimensional reduction. This approach can be considered as the alternative to the global symmetry gauging in the (4,4) extended supergravity with matter, which usually leads to the (classical) scalar potentials unbounded from below [19, 20]. The known exception is the *Wess-Zumino-Novikov-Witten-Liouville*-type (WZNWL-type) *non-linear sigma-model* (NLSM), which reduces to an $SU(2) \times U(1)$ WZNW model in the limit of vanishing Liouville-type interaction [21, 22, 23]. It is precisely the WZNW model whose symmetry gauging amounts to the coupling with the (4,4) supergravitational background in the superconformal gauge [24]. It is of interest to know the full covariant and explicitly supersymmetric form of that NLSM, and the (4,4) superspace supergravity provides the natural framework for that purpose.

Our paper is organized as follows: in sect. 2 the N=4 superspace geometry and the N=4 superfield supergravity constraints are discussed. Sect. 3 is devoted to the component structure of the scalar multiplets TM-I and TM-II. In sect. 3 we briefly review the solution to the N=2 superfield supergravity constraints as presented in ref. [16], which constitutes the pattern we are going to follow to solve the N=4 constraints in the next sect. 4. In sect. 5 we construct the (4,4) locally supersymmetric 2d NLSMs out of TM-I and TM-II matter. In particular, we find the fully covariant (4,4) supersymmetric extension of the Liouville theory, and generate potential terms due to the spontaneous supersymmetry breaking. Our conclusions are summarized in sect. 6. A part of our notation and conventions, as well as some useful identities, are collected in Appendix A. The component structure of the 2d, (4,4) supergravity

multiplet is reviewed in Appendix B. In Appendix C we describe the dimensional reduction of the 4d reduced chiral N=2 superfield down to two dimensions, which generates the scalar potential leading to spontaneous supersymmetry breaking.

2 N=4 superspace geometry

The 2d minimal off-shell (4,4) supergravity in N=4 superspace was first formulated in ref. [14], with the particular 2d, (4,4) hypermultiplet (TM-II) as a scale compensator. There is, in fact, the whole variety of the so-called *variant* representations for a 2d, (4,4) hypermultiplet [25]. The variant representations are inequivalent since there is no way to convert one of them into another while keeping the (4,4) supersymmetry. To distinguish between the different variant representations of the 2d, (4,4) hypermultiplet, we use the classification adopted in ref. [25]. For our purposes in this paper, we only need the two variant off-shell hypermultiplets, TM-I and TM-II. Both have four propagating scalars, which are all singlets in TM-I, while they form one triplet and one singlet in TM-II, with respect to the $SU(2)$ internal symmetry group rotating the N=4 supersymmetry charges [25]. The TM-II is preferable for its use as a (4,4) scale compensator, since it has only one scalar which can represent the usual Weyl transformation parameter. Still, there is no obvious reason against the use of the TM-I as a (4,4) scale compensator, even though it has four physical scalars on equal footing. Since we are not interested in presenting here all possible versions of the N=4 supergravity, we choose its particular version whose superfield strengths form a (4,4) locally supersymmetric TM-I while the (4,4) scale compensator is given by a TM-II, as in ref. [14].

Flat $N = 4$ superspace in two dimensions is parameterised by the coordinates ³

$$z^A = (x^\pm, x^=, \theta^{+i}, \theta^{-i}, \dot{\theta}^+_i, \dot{\theta}^-_i), \quad (2.1)$$

where x^\pm and $x^=$ are two real bosonic (commuting) coordinates, $\theta^{\pm i}$ and their complex conjugates $\dot{\theta}^\pm_i$ are complex fermionic (anticommuting) coordinates, $i = 1, 2$. The fermionic coordinates $\theta^{\pm i}$ are spinors with respect to $SU(2)$. Their complex conjugates were defined by

$$(\theta^{\pm i})^* \equiv -\dot{\theta}^\pm_i, \quad \dot{\theta}^\pm_i = \mathcal{C}^{ij} \theta^\pm_j, \quad (2.2)$$

where the star denotes usual complex conjugation. The $SU(2)$ indices are usually ‘canonically’ contracted from the upper left to the lower right (the North-West/South-East rule), otherwise an extra sign arises. These indices are raised and lowered by

³See Appendix A for more about our notation.

\mathcal{C}^{ij} and \mathcal{C}_{ij} , whose explicit form is given by $\mathcal{C}^{ij} = i\varepsilon^{ij}$ and $(\mathcal{C}^{ij})^* = \mathcal{C}_{ij}$. We prefer to work in a light-cone-type basis, rather than using the 2d gamma matrices (*cf.* refs. [14, 15, 16]).

The spinorial covariant derivatives in the *flat* (4,4) superspace satisfy the algebra

$$\{D_{+i}, D_{+j}^\bullet\} = i \mathcal{C}_{ij} \partial_{\neq}, \quad \{D_{-i}, D_{-j}^\bullet\} = i \mathcal{C}_{ij} \partial_{=} , \quad (2.3)$$

while all other (anti)commutators vanish.

The local symmetries of the N=4 superfield supergravity comprise the N=4 superspace general coordinate transformations, local Lorentz frame rotations and $SU(2)$ internal frame rotations. Therefore, the fully covariant derivatives in the *curved* N=4 superspace should include the tangent space generators for all that symmetries, with the corresponding connections [3, 4]. The superspace geometry of any supergravity theory is described by suitable constraints on the torsion and curvature for the spinorial covariant derivatives. As far as the (4,4) curved superspace is concerned, we define

$$\nabla_A = E_A^M D_M + \Omega_A \mathcal{M} + i \Gamma_A \cdot \mathcal{Y} , \quad (2.4)$$

where the N=4 supervielbein E_A^M , the Lorentz generator \mathcal{M} with the Lorentz connection Ω_A , and the $SU(2)$ generators \mathcal{Y}_i^j with the $SU(2)$ connection $(\Gamma_A)_j^i$ have been introduced. We sometimes use the dot product, $\Gamma_A \cdot \mathcal{Y} \equiv (\Gamma_A)_j^i \mathcal{Y}_i^j$, in order to simplify our notation. The operators ∇_A change covariantly under all the local symmetry transformations by definition, i.e.

$$\nabla'_A = e^{-\mathcal{H}} \nabla_A e^{\mathcal{H}} , \quad \mathcal{H} = H^M \partial_M + H \mathcal{M} + i H_j^i \mathcal{Y}_i^j , \quad (2.5)$$

where H^M , H , and $i H_j^i$ are the infinitesimal superfield parameters for the N=4 superspace general coordinate, local Lorentz and $SU(2)$ transformations, respectively.

We assume that the supervielbein is invertible, and identify the lowest-order component in the θ -expansion of the superfield E_μ^a with the zweibein, $E_\mu^a| = e_\mu^a$. Similarly, $E_\mu^{i\pm}| = \psi_\mu^{i\pm}$ and $\Gamma_{\mu j}^i| = A_{\mu j}^i$ define the rest of the gauge fields for the 2d conformal (4,4) supergravity. The superfield torsion and curvature tensors are defined as usual, namely

$$[\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + R_{AB} \mathcal{M} + i F_{AB} \cdot \mathcal{Y} . \quad (2.6)$$

The generators for the local Lorentz and $SU(2)$ frame transformations are defined by their action on spinors or the spinorial derivatives,

$$[\mathcal{M}, \nabla_{\pm i}] = \pm \frac{1}{2} \nabla_{\pm i} , \quad [\mathcal{M}, \nabla_{\pm}^\bullet] = \pm \frac{1}{2} \nabla_{\pm}^\bullet ,$$

$$\begin{aligned}
[\mathcal{Y}_i^j, \nabla_{\pm k}] &= +\delta_k^j \nabla_{\pm i} - \frac{1}{2} \delta_i^j \nabla_{\pm k} , \\
[\mathcal{Y}_i^j, \nabla_{\pm}^k] &= -\delta_i^k \nabla_{\pm}^j + \frac{1}{2} \delta_i^j \nabla_{\pm}^k .
\end{aligned} \tag{2.7}$$

The supervielbein and superconnections define a highly reducible representation of N=4 supersymmetry, and they have therefore to be restricted by covariant constraints [26].

The constraints defining the N=4 superfield supergravity are given by (*cf.* ref. [14])

$$\begin{aligned}
\{\nabla_{\pm i}, \nabla_{\pm j}\} &= 0 , \quad \{\nabla_{+i}, \nabla_{+j}^{\bullet}\} = i\mathcal{C}_{ij} \nabla_{\neq} , \quad \{\nabla_{-i}, \nabla_{-j}^{\bullet}\} = i\mathcal{C}_{ij} \nabla_{=} , \\
\{\nabla_{+i}, \nabla_{-}^j\} &= -\frac{i}{2} R^* \left(\delta_i^j \mathcal{M} - \mathcal{Y}_i^j \right) , \\
\{\nabla_{+i}, \nabla_{-}^{\bullet j}\} &= -\frac{i}{2} S \left(\delta_i^j \mathcal{M} - \mathcal{Y}_i^j \right) - \frac{1}{2} T \left(\delta_i^j \mathcal{M} - \mathcal{Y}_i^j \right) ,
\end{aligned} \tag{2.8}$$

as well as their complex conjugates. Given the constraints above, the additional constraints on the (4,4) supergravity superfield strengths R_{AB} and $(F_{AB})_j^i$ follow from the Bianchi identities. For instance, the Bianchi identity for the torsion $T^A = \nabla E^A$ reads

$$\nabla T^A = E^B R_B^A . \tag{2.9}$$

As far as the full algebra of the covariant derivatives is concerned, we find

$$\begin{aligned}
[\nabla_{+i}, \nabla_{=}] &= \frac{1}{4} \left[-2R^* \nabla_{-i}^{\bullet} + 2(S - iT) \nabla_{-i} \right. \\
&\quad \left. + \left(\nabla_{-}^{\bullet j} R^* - \nabla_{-}^j (S - iT) \right) (\mathcal{C}_{ij} \mathcal{M} - \mathcal{Y}_{ij}) \right] , \\
[\nabla_{-i}, \nabla_{\neq}] &= \frac{1}{4} \left[-2R^* \nabla_{+i}^{\bullet} - 2(S + iT) \nabla_{+i} \right. \\
&\quad \left. - \left(\nabla_{+}^{\bullet j} R^* - \nabla_{+}^j (S + iT) \right) (\mathcal{C}_{ij} \mathcal{M} + \mathcal{Y}_{ij}) \right] , \\
[\nabla_{+i}, \nabla_{\neq}] &= 0 , \quad [\nabla_{-i}, \nabla_{=}] = 0 , \\
[\nabla_{\neq}, \nabla_{=}] &= \frac{i}{4} \left[+(\nabla_{-}^i R) \nabla_{+i} - (\nabla_{+}^i R) \nabla_{-i} - (\nabla_{-}^{\bullet i} R^*) \nabla_{+i}^{\bullet} + (\nabla_{+}^{\bullet i} R^*) \nabla_{-i}^{\bullet} \right. \\
&\quad - (\nabla_{-}^{\bullet i} (S + iT)) \nabla_{+i} + (\nabla_{-}^i (S - iT)) \nabla_{+i}^{\bullet} \\
&\quad \left. - (\nabla_{+}^{\bullet i} (S - iT)) \nabla_{-i} + (\nabla_{+}^i (S + iT)) \nabla_{-i}^{\bullet} \right] \\
&\quad + \frac{1}{2} \left[RR^* - S^2 - T^2 - \frac{i}{4} (\nabla_{+}^i \nabla_{-i} R) + \frac{i}{4} (\nabla_{+}^{\bullet i} \nabla_{-i}^{\bullet} R^*) \right. \\
&\quad \left. - \frac{i}{4} (\nabla_{+}^{\bullet i} \nabla_{-i} (S - iT)) + \frac{i}{4} (\nabla_{+}^i \nabla_{-i}^{\bullet} (S + iT)) \right] \mathcal{M} \\
&\quad + \frac{i}{8} \left[(\nabla_{+}^i \nabla_{-j} R) - (\nabla_{+}^{\bullet i} \nabla_{-j}^{\bullet} R^*) \right. \\
&\quad \left. + (\nabla_{+}^{\bullet i} \nabla_{-j} (S - iT)) - (\nabla_{+}^i \nabla_{-j}^{\bullet} (S + iT)) \right] \mathcal{Y}_i^j ,
\end{aligned}$$

$$[\nabla_{\mp}, \nabla_{\mp}] = 0, \quad [\nabla_{=}, \nabla_{=}] = 0. \quad (2.10)$$

The constraints following from the Bianchi identity

$$\nabla F = 0 \quad (2.11)$$

for the $SU(2)$ superfield strength F are given by

$$\begin{aligned} \nabla_{+k} F_{+i -j} + \nabla_{+j} F_{+k -i} &= 0, \\ \nabla_{-k} F_{+i -j} + \nabla_{-i} F_{+j -k} &= 0, \\ \nabla_{+k} F_{+i -j}^{\bullet} + \nabla_{+j} F_{+k -i}^{\bullet} &= 0, \\ \nabla_{-k} F_{+i -j}^{\bullet} + \nabla_{-i} F_{+j -k}^{\bullet} &= 0. \end{aligned} \quad (2.12)$$

The defining constraints (2.8) also imply further consistency relations having the form

$$\begin{aligned} \nabla_{-k}^{\bullet} F_{+i -j} + \nabla_{-i} F_{+j -k}^{\bullet} - T_{-i}^{\bullet} F_{=+j}^{\bullet} &= 0, \\ \nabla_{+k}^{\bullet} F_{+i -j} + \nabla_{+j} F_{+k -i}^{\bullet} - T_{+j}^{\bullet} F_{\mp -i}^{\bullet} &= 0. \end{aligned} \quad (2.13)$$

Taken together, they lead to the certain constraints on the (4,4) supergravity field strengths which comprise the complex scalar superfield R and the two real ones, S and T . We find

$$\begin{aligned} \nabla_{\pm i}^{\bullet} R &= 0, \quad \nabla_{\pm i} R = \pm 2 \nabla_{\pm i}^{\bullet} S, \quad \nabla_{\pm i} S = \pm i \nabla_{\pm i} T, \\ [\nabla_{+}^i, \nabla_{+i}] R &= [\nabla_{-}^i, \nabla_{-i}] R = 0, \end{aligned} \quad (2.14)$$

and their conjugates, where the signs are correlated, as well as the additional reality condition

$$\left(\nabla_{+i} \nabla_{-j}^{\bullet} S \right)^* = \nabla_{+}^i \nabla_{-}^{\bullet j} S, \quad (2.15)$$

Eqs. (2.14) and (2.15) define the *twisted*-I hypermultiplet (TM-I). It is not difficult to check that the TM-I has $8_{\mathbf{B}} \oplus 8_{\mathbf{F}}$ independent off-shell degrees of freedom (see also the next sect. 3).

Some of the constraints given above were also found in ref. [14]. In particular, it is straightforward to verify that no more consistency relations follow from the Bianchi identities. The (4,4) supergravity multiplet, comprising a graviton e_{μ}^a , four gravitini $\psi_{\mu}^{i\pm}$, an $SU(2)$ triplet of graviphotons A_{μ}^I , $I = 1, 2, 3$, a complex scalar R , and two real scalars S and T , appears at the component level. The supersymmetry transformation laws for the components of the (4,4) conformal supergravity multiplet are collected in Appendix B.

3 TM-I and TM-II in curved (4,4) superspace

In this section we provide the superspace formulation for two off-shell (4,4) hypermultiplets, TM-I and TM-II, in the presence of the (4,4) supergravity. The rigid (4,4) supersymmetry hypermultiplets are known for a long time (see, e.g., ref. [25] for a recent review). Their minimal versions, TM-I and TM-II, each have $\delta_{\mathbf{B}} \oplus \delta_{\mathbf{F}}$ independent components.

The constraints defining the TM-I were already given in the preceding sect. 2, namely,

$$\begin{aligned} \nabla_{\pm i}^{\bullet} B &= 0, \quad \nabla_{\pm i} B = \pm 2 \nabla_{\pm i}^{\bullet} F, \quad \nabla_{\pm i} F = \pm i \nabla_{\pm i} G, \\ [\nabla_+^i, \nabla_{+i}] B &= [\nabla_-^i, \nabla_{-i}] B = 0, \end{aligned} \quad (3.1)$$

and their conjugates, in terms of four scalar superfields, the complex one B and two real ones F and G , with the additional reality condition

$$\left(\nabla_{+i}^{\bullet} \nabla_{-j}^{\bullet} F \right)^* = \nabla_+^i \nabla_-^j F. \quad (3.2)$$

The independent components of the TM-I can be chosen as follows:

$$\dim - 0 : B, \quad B^*, \quad F, \quad G, \quad (4_{\mathbf{B}})$$

$$\dim - \frac{1}{2} : \nabla_{\pm i} F = \lambda_{\pm i} \quad \text{and} \quad \bar{\lambda}_{\pm i}^{\bullet}, \quad (8_{\mathbf{F}}) \quad (3.3)$$

$$\dim - 1 : \nabla_{+i}^{\bullet} \lambda_{-j} = A_{ij} \equiv A_{(ij)} + \mathcal{C}_{ij} A, \quad (4_{\mathbf{B}})$$

where $\{A_{(ij)}\}^* = A^{(ij)}$ and A is real. It is now straightforward to determine the supersymmetry transformation laws for the TM-I components from eqs. (2.8) and (3.1)–(3.3). We find in addition that

$$\begin{aligned} \nabla_{+i} \lambda_{+j} &= \frac{i}{2} \mathcal{C}_{ij} \nabla_{\neq} B^*, \quad \nabla_{-i} \lambda_{+j} = 0, \\ \nabla_{+i}^{\bullet} \lambda_{+j} &= -\frac{i}{2} \mathcal{C}_{ij} \nabla_{\neq} F, \quad \nabla_{-i}^{\bullet} \lambda_{+j} = -A_{ji}, \\ \nabla_{-i} \lambda_{-j} &= -\frac{i}{2} \mathcal{C}_{ij} \nabla_{\neq} B^*, \quad \nabla_{+i} \lambda_{-j} = 0, \\ \nabla_{-i}^{\bullet} \lambda_{-j} &= -\frac{i}{2} \mathcal{C}_{ij} \nabla_{\neq} F, \quad \nabla_{+i}^{\bullet} \lambda_{-j} = A_{ij}, \\ \nabla_{+k} A &= \frac{1}{2} \nabla_{\neq} \lambda_{-k} + \frac{1}{4} (S - iT) \lambda_{+k}, \quad \nabla_{-k} A = \frac{1}{2} \nabla_{\neq} \lambda_{+k} + \frac{1}{4} (S + iT) \lambda_{-k}, \\ \nabla_{+k} A_{(ij)} &= i \mathcal{C}_{k(i} \nabla_{\neq} \lambda_{-j)} - \frac{i}{2} (S - iT) \mathcal{C}_{k(i} \lambda_{+j)}, \\ \nabla_{-k} A_{(ij)} &= -i \mathcal{C}_{k(i} \nabla_{\neq} \lambda_{+j)} - \frac{i}{2} (S + iT) \mathcal{C}_{k(i} \lambda_{-j)}. \end{aligned} \quad (3.4)$$

Together with their complex conjugates and the defining equations it completes the list of the (4,4) local supersymmetry transformation rules for the TM-I components.

The TM-I is not the only minimal off-shell (4,4) hypermultiplet known to exist in two dimensions. The different minimal hypermultiplet, TM-II, can be most naturally introduced after noticing that the defining constraints (2.8) of the 2d, (4,4) supergravity have additional local symmetry. Namely, they are invariant under the (4,4) super-Weyl transformations (*cf.* the super-Weyl symmetry of the simple ($N = 1$, 2d) superfield supergravity [27]):

$$\delta \nabla_{\pm i} = \frac{1}{2} P \nabla_{\pm i} + L_i^j \nabla_{\pm j} \mp (\nabla_{\pm i} P) \mathcal{M} \pm i (\nabla_{\pm j} P) \mathcal{Y}_i^j , \quad (3.5)$$

and

$$\delta R = P R , \quad \delta(S - iT) = P(S - iT) , \quad (3.6)$$

where the infinitesimal (4,4) superfield parameters P and L_i^j satisfy the constraints

$$\begin{aligned} \nabla_{\pm k} L_{ij} &= \pm \frac{i}{2} (\mathcal{C}_{ik} \nabla_{\pm j} + \mathcal{C}_{jk} \nabla_{\pm i}) P , \\ [\nabla_+^{\bullet i}, \nabla_+^j] L_{ij} &= [\nabla_-^{\bullet i}, \nabla_-^j] L_{ij} = 0 . \end{aligned} \quad (3.7)$$

In eq. (3.7) the signs are correlated, $L_{ij} = L_{ji}$ or $L_i^i = 0$, and the following reality conditions are imposed:

$$(L_{ij})^* = L^{ij} , \quad P^* = P . \quad (3.8)$$

Eqs. (3.7) and (3.8) define the *twisted-II* (TM-II) hypermultiplet in the (4,4) superspace [25]. The independent components of the TM-II can be chosen as follows:

$$\begin{aligned} \dim - 0 : \quad & P , \quad L_i^j , \quad (4_B) \\ \dim - \frac{1}{2} : \quad & \pm \frac{3i}{2} \nabla_{\pm i} P = \chi_{\pm i} \quad \text{and} \quad \bar{\chi}_{\pm i}^{\bullet} , \quad (8_F) \\ \dim - 1 : \quad & \nabla_+^i \nabla_-^j L_{ij} = U , \quad \text{and} \quad \nabla_+^{\bullet i} \nabla_-^j L_{ij} = U^* , \\ & \nabla_+^{\bullet i} \nabla_-^j L_{ij} = M + iN , \quad \text{and} \quad \nabla_+^i \nabla_-^{\bullet j} L_{ij} = M - iN , \quad (4_B) \end{aligned} \quad (3.9)$$

where the M and N fields are real. It is straightforward to determine the rest of the supersymmetry transformation laws for the TM-II components. We find

$$\begin{aligned} \nabla_{+i} \chi_{+j} &= 0 , \quad \nabla_{-i} \chi_{-j} = 0 , \\ \nabla_{+i} \chi_{-j} &= \frac{3i}{4} R^* L_{ij} - \frac{1}{2} \mathcal{C}_{ij} U , \quad \nabla_{-i} \chi_{+j} = \frac{3i}{4} R^* L_{ij} + \frac{1}{2} \mathcal{C}_{ij} U , \\ \nabla_{+i}^{\bullet} \chi_{+j} &= -\frac{3i}{2} \nabla_{+} L_{ij} + \frac{3}{4} \mathcal{C}_{ij} \nabla_{+} P , \quad \nabla_{-i}^{\bullet} \chi_{-j} = -\frac{3i}{2} \nabla_{-} L_{ij} + \frac{3}{4} \mathcal{C}_{ij} \nabla_{-} P , \end{aligned}$$

$$\begin{aligned}
\nabla_{-i}^\bullet \chi_{+j} &= \frac{3i}{4}(S - iT)L_{ij} + \frac{1}{2}\mathcal{C}_{ij}(M - iN) , \\
\nabla_{+i}^\bullet \chi_{-j} &= -\frac{3i}{4}(S + iT)L_{ij} - \frac{1}{2}\mathcal{C}_{ij}(M + iN) ,
\end{aligned} \tag{3.10a}$$

and

$$\begin{aligned}
\nabla_{+i} U &= -\frac{i}{2}R^* \chi_{+i} , \quad \nabla_{-i} U = \frac{i}{2}R^* \chi_{-i} , \\
\nabla_{+i}^\bullet U &= -\frac{3i}{2}\nabla_{\neq} \chi_{-i} + 3i(\nabla_+^j S)L_{ij} + \frac{5i}{4}(S + iT)\chi_{+i} + \frac{3i}{4}R^* \chi_{+i}^\bullet , \\
\nabla_{-i}^\bullet U &= -\frac{3i}{2}\nabla_{=} \chi_{+i} + 3i(\nabla_-^j S)L_{ij} + \frac{5i}{4}(S - iT)\chi_{-i} - \frac{3i}{4}R^* \chi_{-i}^\bullet ,
\end{aligned}$$

$$\begin{aligned}
\nabla_{+i}(M + iN) &= \frac{3i}{2}\nabla_{\neq} \chi_{-i} - 3i(\nabla_+^j S)L_{ij} - \frac{5i}{4}R^* \chi_{+i} - \frac{3i}{4}(S + iT)\chi_{+i} , \\
\nabla_{-i}(M - iN) &= \frac{3i}{2}\nabla_{=} \chi_{+i} - 3i(\nabla_-^j S)L_{ij} + \frac{5i}{4}R^* \chi_{-i} - \frac{3i}{4}(S - iT)\chi_{-i} , \\
\nabla_{+i}(M - iN) &= -\frac{i}{2}(S - iT)\chi_{+i} , \\
\nabla_{-i}(M + iN) &= -\frac{i}{2}(S + iT)\chi_{-i} ,
\end{aligned} \tag{3.10b}$$

Together with their complex conjugates and the defining equations it completes the list of the (4,4) local supersymmetry transformation rules for the TM-II components.

4 The (2,2) supergravity solution

In this section, we briefly review some aspects of the (2,2) extended 2d supergravity in N=2 superspace, and its solution as presented in ref. [16], which are going to be relevant for our (4,4) supersymmetric construction in the next section.

The N=2 superspace has two real bosonic coordinates x^\pm and x^\pm , and two complex fermionic coordinates θ^+ and θ^- , as well as their conjugates $\dot{\theta}^+$ and $\dot{\theta}^-$. In addition to the N=2 superspace general coordinate transformations, the full local symmetries of the nonminimal (2,2) supergravity include the local Lorentz symmetry, an axial $U_A(1)$ and a vector $U_V(1)$ internal symmetries.

The geometry of the (2,2) superfield supergravity is described in terms of the covariant spinorial derivatives

$$\nabla_\pm = E_\pm^M \partial_M + \Omega_\pm \mathcal{M} + \Gamma_\pm \mathcal{X} + \tilde{\Gamma}_\pm \tilde{\mathcal{X}} , \tag{4.1}$$

where the generators of the local Lorentz, $U_V(1)$ and $U_A(1)$ symmetries, \mathcal{M} , \mathcal{X} and $\tilde{\mathcal{X}}$, respectively, have been introduced.

The nonminimal (2,2) superfield supergravity is defined by the constraints (*cf.* ref. [28])

$$\begin{aligned}
\{\nabla_{\pm}, \nabla_{\pm}\} &= 0, \quad \{\nabla_{+}, \nabla_{+}^{\bullet}\} = i\nabla_{\#}, \quad \{\nabla_{-}, \nabla_{-}^{\bullet}\} = i\nabla_{=}, \\
\{\nabla_{+}, \nabla_{-}\} &= -\frac{1}{2}R^{*}(\mathcal{M} - i\mathcal{X}), \\
\{\nabla_{+}, \nabla_{-}^{\bullet}\} &= -F(\mathcal{M} - i\tilde{\mathcal{X}}).
\end{aligned} \tag{4.2}$$

The *minimal* N=2 supergravities appear under the restriction $F = 0$ or $R = 0$ [28].

The constraints of eq. (4.2) are invariant under the additional local Weyl (scale) transformations in N=2 superspace [28, 29],

$$\begin{aligned}
E_{\pm} &\rightarrow e^L E_{\pm}, \quad \Omega_{\pm} \rightarrow e^L (\Omega_{\pm} \pm 2E_{\pm}L), \\
\Gamma_{\pm} &\rightarrow e^L (\Gamma_{\pm} \mp i2E_{\pm}L), \quad \tilde{\Gamma}_{\pm} \rightarrow e^L (\tilde{\Gamma}_{\pm} - i2E_{\pm}L), \\
R^{*} &\rightarrow e^{2L} (R^{*} + 4[\nabla_{-}, \nabla_{+}]L), \quad F \rightarrow e^{2L} (F - 2i[\nabla_{-}^{\bullet}, \nabla_{+}]L),
\end{aligned} \tag{4.3}$$

where the Weyl superfield parameter L can be restricted to be real (its imaginary part can be absorbed by the local $U_V(1)$ transformations).

To solve the constraints (4.2), Grisaru and Wehlau [16] first removed many irrelevant superfields by imposing the supersymmetric gauge

$$E_{+} = e^{S^{*}} (\hat{E}_{+} + A_{+}^{-} \hat{E}_{-}), \quad E_{-} = e^{S^{*}} (\hat{E}_{-} + A_{-}^{+} \hat{E}_{+}), \tag{4.4}$$

which does not introduce propagating ghosts. In eq. (4.4), the reduced differential operators

$$\hat{E}_{\pm} = e^{-iH^m \partial_m} D_{\pm} e^{+iH^m \partial_m} \equiv D_{\pm} + iH_{\pm}^m \partial_m, \tag{4.5}$$

a real vector superfield H^m and a complex scalar superfield S have been introduced. Substituting eq. (4.4) into the constraints (4.2), one finds that the superfield connections A_{+}^{-} and A_{-}^{+} satisfy the *algebraic* (quadratic) equations, which determine them as the functions of H^m . The two remaining independent superfields H^m and S are just the (2,2) prepotentials of the non-minimal theory. In particular, the superfield S can be recognized as the N=2 scale compensator since the N=2 Weyl transformation is equivalent to a shift in S [16]. In the minimal versions of the (2,2) supergravity, the scale compensator is either a chiral or a twisted chiral N=2 scalar superfield [14, 16, 28, 29]. It should be noticed that the supersymmetric gauge-choice in eq. (4.4) is *not* symmetric with respect to an exchange of $(-)$ and (\bullet) objects [17], so that one should not expect that the two minimal versions appear on equal footing from the non-minimal theory.

5 Towards a solution to the (4,4) supergravity constraints

The natural (4,4) supersymmetric generalisation of the flat (2,2) superdifferential operators in eq. (4.5) is given by

$$\hat{E}_{\pm i} = e^{-\mathcal{H}} D_{\pm i} e^{\mathcal{H}} , \quad (5.1)$$

where the operator \mathcal{H} of eq. (2.5) can be restricted to have only the ‘space-time’ imaginary part, $\mathcal{H} = iH^m \partial_m$, with a real vector superfield H^m , by making certain supersymmetric gauge choices which do not lead to propagating ghosts. The operators $\hat{E}_{\pm i}$, their conjugates $\hat{E}_{\pm i}^\bullet$, and $\hat{E}_{\mp,=}$ to be defined below (see eq. (5.3)) form a convenient linearly independent basis of derivative operators, and satisfy a closed algebra in superspace,

$$\begin{aligned} \{\hat{E}_{+i}, \hat{E}_{-}^{\bullet j}\} &= \hat{G}_{+i-}^{\bullet j\neq} \hat{E}_{\neq} + \hat{G}_{+i-}^{\bullet j=} \hat{E}_{=} , \\ \{\hat{E}_{-i}, \hat{E}_{+}^{\bullet j}\} &= \hat{G}_{-i+}^{\bullet j\neq} \hat{E}_{\neq} + \hat{G}_{-i+}^{\bullet j=} \hat{E}_{=} , \end{aligned} \quad (5.2)$$

where we have introduced

$$\hat{E}_{\neq} \equiv \frac{i}{2} \{\hat{E}_{+}^{\bullet i}, \hat{E}_{+i}^{\bullet}\} , \quad \text{and} \quad \hat{E}_{=} \equiv \frac{i}{2} \{\hat{E}_{-}^{\bullet i}, \hat{E}_{-i}^{\bullet}\} . \quad (5.3)$$

The ‘structure constants’ \hat{G} ’s in eq. (5.2) are actually certain functions of \mathcal{H} (or H^m , after gauge-fixing), whose explicit form is determined by eq. (5.1). The full supervielbein operators should be related to that of eq. (5.1), in accordance with the Frobenius theorem [3, 4], as

$$\begin{aligned} E_{+i} &= (K_1)_i^j \left[\hat{E}_{+j} + A_{+j}^{-l} \hat{E}_{-l} \right] , \\ E_{-i} &= (K_2)_i^j \left[\hat{E}_{-j} + A_{-j}^{+l} \hat{E}_{+l} \right] , \end{aligned} \quad (5.4)$$

where the scalar $SU(2)$ -tensor superfields $(K_{1,2})_i^j$ and the vector $SU(2)$ -tensor superfields $(A_a)_i^j = (A_{+i+}^j, A_{-i-}^j)$ have been introduced. We have in fact assumed in eq. (5.4) that the generalised ‘holomorphicity’ takes place which allows only ‘undotted’ indices to appear, like in the (2,2) case. The equations for the spinorial supervielbein operators with ‘dotted’ indices are formally obtained from eq. (5.4) by complex conjugation.

We now want to make use of the already established fact (sect. 3) that the two-dimensional (4,4) superspace supergravity defined by the constraints (2.8) is superconformally flat, similarly to the $N = 1$ and $N = 2$ superspace supergravities in two

dimensions [27, 28]. It implies that the relation between the flat and curved spinorial derivatives, as written in eq. (5.4), should take the form of an (4,4) superconformal transformation. In sect. 3 we found the infinitesimal form of the (4,4) super-Weyl transformation but, in order to specify the matrices $K_{1,2}$ in eq. (5.4), we need its finite form. As regards the (4,4) super-Weyl transformation law for the spinorial supervielbein components, one easily finds that

$$(K_1)_i{}^j = (K_2)_i{}^j \equiv K_i{}^j = \exp(P\mathbf{1} + L)_i{}^j, \quad (5.5)$$

where the P and $L_i{}^j$ superfield parameters (forming a TM-II) have been introduced in eqs. (3.7) and (3.8), and $\mathbf{1}$ is a unit matrix.

It is straightforward to substitute our Ansatz (5.2) and (5.3) into the constraints (2.8). As a result, all the superconnections in eq. (2.4), as well as the newly introduced superfields A 's, are unambiguously determined, as we are now going to demonstrate.

First, using the constraint $\{\nabla_{+i}, \nabla_{+j}\} = 0$, we find

$$\begin{aligned} E_{+i}E_{+j} + \frac{1}{2}\Omega_{+i}E_{+j} + i\Gamma_{+i}{}^k{}^l(\delta_j{}^k E_{+l} - \frac{1}{2}\delta_l{}^k E_{+j}) + (i \leftrightarrow j) &= 0, \\ E_{+i}\Omega_{+j} + i\Gamma_{+i}{}^k{}^l(\delta_j{}^k \Omega_{+l} - \frac{1}{2}\delta_l{}^k \Omega_{+j}) + (i \leftrightarrow j) &= 0, \\ iE_{+i}\Gamma_{+j}\mathcal{Y} + \frac{i}{2}\Omega_{+i}\Gamma_{+j}\mathcal{Y} - \Gamma_{+i}{}^k{}^l(\delta_j{}^k \Gamma_{+l}\mathcal{Y} - \frac{1}{2}\delta_l{}^k \Gamma_{+j}\mathcal{Y}) - \\ - \Gamma_{+i}{}^r{}^l\Gamma_{+j}{}^s{}^r\mathcal{Y}_s + (i \leftrightarrow j) &= 0. \end{aligned}$$

Since $\hat{E}_{\pm i}$ are linearly independent, it yields

$$\frac{1}{2}(\Omega_{+i}\delta_j{}^l + 2i\Gamma_{+i}{}^j{}^l)K_l{}^m + (i \leftrightarrow j) = -K_i{}^k(\hat{E}_{+k}K_j{}^m + A_{+k}{}^{-l}\hat{E}_{-l}K_j{}^m) + (i \leftrightarrow j), \quad (5.6)$$

while $A_{+i}{}^{-j}$ must satisfy a differential equation

$$\hat{E}_{+k}A_{+m}{}^{-n} + A_{+k}{}^{-l}(\hat{E}_{-l}A_{+m}{}^{-n}) = 0. \quad (5.7)$$

By using the equation $\{\nabla_{-i}, \nabla_{-j}\} = 0$, we similarly get

$$\begin{aligned} E_{-i}E_{-j} - \frac{1}{2}\Omega_{-i}E_{-j} + i\Gamma_{-i}{}^k{}^l(\delta_j{}^k E_{-l} - \frac{1}{2}\delta_l{}^k E_{-j}) + (i \leftrightarrow j) &= 0, \\ E_{-i}\Omega_{-j} + i\Gamma_{-i}{}^k{}^l(\delta_j{}^k \Omega_{-l} - \frac{1}{2}\delta_l{}^k \Omega_{-j}) + (i \leftrightarrow j) &= 0, \\ iE_{-i}\Gamma_{-j}\mathcal{Y} - \frac{i}{2}\Omega_{-i}\Gamma_{-j}\mathcal{Y} - \Gamma_{-i}{}^k{}^l(\delta_j{}^k \Gamma_{-l}\mathcal{Y} - \frac{1}{2}\delta_l{}^k \Gamma_{-j}\mathcal{Y}) - \\ - \Gamma_{-i}{}^r{}^l\Gamma_{-j}{}^s{}^r\mathcal{Y}_s + (i \leftrightarrow j) &= 0. \end{aligned}$$

It implies

$$\frac{1}{2}(\Omega_{-i}\delta_j{}^l - 2i\Gamma_{-i}{}^j{}^l)K_l{}^m + (i \leftrightarrow j) = K_i{}^k(\hat{E}_{-k}K_j{}^m + A_{-k}{}^{+l}\hat{E}_{+l}K_j{}^m) + (i \leftrightarrow j), \quad (5.8)$$

while A_{-i}^{+j} must satisfy an equation

$$\hat{E}_{-k}A_{-m}^{+n} + A_{-k}^{+l}(\hat{E}_{+l}A_{-m}^{+n}) = 0 . \quad (5.9)$$

The next constraint $\{\nabla_{+i}, \nabla_{-j}\} = -\frac{i}{2} \bar{R} \left(\delta_i^j \mathcal{M} - \mathcal{Y}_i^j \right)$ yields

$$\begin{aligned} & \{E_{+i}, E_{-j}\} - \frac{1}{2} \Omega_{+i} E_{-j} + \frac{1}{2} \Omega_{-j} E_{+i} + \\ & + i \Gamma_{+i}{}^k{}_l \left(\delta_j^k E_{-l} - \frac{1}{2} \delta_l^k E_{-j} \right) + i \Gamma_{-j}{}^k{}_l \left(\delta_i^k E_{+l} - \frac{1}{2} \delta_l^k E_{+i} \right) = 0 , \\ & E_{+i} \Omega_{-j} + E_{-j} \Omega_{+i} - \Omega_{+i} \Omega_{-j} + \\ & + i \Gamma_{+i}{}^k{}_l \left(\delta_j^k \Omega_{-l} - \frac{1}{2} \delta_l^k \Omega_{-j} \right) + i \Gamma_{-j}{}^k{}_l \left(\delta_i^k \Omega_{+l} - \frac{1}{2} \delta_l^k \Omega_{+i} \right) = -\frac{i}{2} \bar{R} \mathcal{C}_{ij} , \\ & i E_{+i} \Gamma_{-j} \mathcal{Y} + i E_{-j} \Gamma_{+i} \mathcal{Y} - \frac{i}{2} \Omega_{+i} \Gamma_{-j} \mathcal{Y} + \frac{i}{2} \Omega_{-j} \Gamma_{+i} \mathcal{Y} - \\ & - \Gamma_{+i}{}^k{}_l \left(\delta_j^k \Gamma_{-l} \mathcal{Y} - \frac{1}{2} \delta_l^k \Gamma_{-j} \mathcal{Y} \right) - \Gamma_{-j}{}^k{}_l \left(\delta_i^k \Gamma_{+l} \mathcal{Y} - \frac{1}{2} \delta_l^k \Gamma_{+i} \mathcal{Y} \right) - \\ & - \Gamma_{+i}{}^r{}_l \Gamma_{-j}{}^l{}_s \mathcal{Y}_s{}^r - \Gamma_{-j}{}^r{}_l \Gamma_{+i}{}^l{}_s \mathcal{Y}_s{}^r = \frac{i}{2} \bar{R} \mathcal{Y}_{ij} . \end{aligned}$$

Using eqs. (5.6) and (5.9), we find after some algebra that

$$\begin{aligned} \Omega_{+i} &= +K_i^k (\hat{E}_{-n} A_{+k}^{-n} - A_{+k}^{-m} \hat{E}_{+r} A_{-m}^{+r}) , \\ \Omega_{-i} &= -K_i^k (\hat{E}_{+n} A_{-k}^{+n} - A_{-k}^{+n} \hat{E}_{-r} A_{+n}^{-r}) . \end{aligned} \quad (5.10)$$

We are now in a position to calculate the connections $\Gamma_{\pm i}{}^l{}_j$. In the matrix form, they are given by

$$\begin{aligned} \Gamma_{+i} &= \frac{i}{2} \Omega_{+i} + \frac{i}{2} E_{+i} P + i e^L U_{+i} e^{-L} , \\ \Gamma_{-i} &= \frac{i}{2} \Omega_{-i} - \frac{i}{2} E_{-i} P - i e^L U_{-i} e^{-L} , \end{aligned} \quad (5.11)$$

where we have defined

$$\begin{aligned} U_{+i}{}^k{}_j &= (e^{\frac{1}{2}P+L})_j{}^l (\hat{U}_{+il}{}^k + A_{+l}^{-m} \hat{U}_{-i}{}^m{}_k) , \\ U_{-i}{}^k{}_j &= (e^{\frac{1}{2}P+L})_j{}^l (\hat{U}_{-il}{}^k + A_{-l}^{+m} \hat{U}_{+i}{}^m{}_k) . \end{aligned} \quad (5.12)$$

To get eq. (5.11), we used the identity

$$\hat{E}_{\pm i} (e^L)_j{}^k = (e^L)_j{}^r (\hat{U}_{\pm i})_r{}^k . \quad (5.13)$$

and the matrix relation

$$\hat{E}_{\pm i} e^{\frac{1}{2}P+L} = \frac{1}{2} (\hat{E}_{\pm i} P) e^{\frac{1}{2}P+L} + e^{\frac{1}{2}P} e^L \hat{U}_{\pm i} . \quad (5.14)$$

The constraint $\{\nabla_{+i}, \nabla_{-j}\} = -\frac{i}{2} (S - iT) \left(\delta_i^j \mathcal{M} - \mathcal{Y}_i^j \right)$ is equivalent to

$$\{E_{+i}, E_{-j}\} - \frac{1}{2} \Omega_{+i} E_{-j} + \frac{1}{2} \Omega_{-j} E_{+i} -$$

$$\begin{aligned}
& -i \Gamma_{+i k}{}^l \left(\delta_l^j E_{-}^k - \frac{1}{2} \delta_l^k E_{-}^j \right) - i \Gamma_{-k}{}^j{}^l \left(\delta_i^k E_{+l} - \frac{1}{2} \delta_l^k E_{+i} \right) = 0 , \\
& E_{+i} \Omega_{-}^j + E_{-}^j \Omega_{+i} - \Omega_{+i} \Omega_{-}^j - \\
& -i \Gamma_{+i k}{}^l \left(\delta_l^j \Omega_{-}^k - \frac{1}{2} \delta_l^k \Omega_{-}^j \right) - i \Gamma_{-k}{}^j{}^l \left(\delta_i^k \Omega_{+l} - \frac{1}{2} \delta_l^k \Omega_{+i} \right) = -\frac{i}{2} (S - iT) \delta_i^j , \\
& i E_{+i} \Gamma_{-}^j \mathcal{Y} - i E_{-}^j \Gamma_{+i} \mathcal{Y} - \frac{i}{2} \Omega_{+i} \Gamma_{-}^j \mathcal{Y} - \frac{i}{2} \Omega_{-}^j \Gamma_{+i} \mathcal{Y} + \\
& + \Gamma_{+i k}{}^l \left(\delta_l^j \Gamma_{-}^k \mathcal{Y} - \frac{1}{2} \delta_l^k \Gamma_{-}^j \mathcal{Y} \right) - \Gamma_{-k}{}^j{}^l \left(\delta_i^k \Gamma_{+l} \mathcal{Y} - \frac{1}{2} \delta_l^k \Gamma_{+i} \mathcal{Y} \right) - \\
& - \Gamma_{+i r}{}^l \Gamma_{-l}^j{}^s \mathcal{Y}_s^r - \Gamma_{-r}^j{}^l \Gamma_{+i l}{}^s \mathcal{Y}_s^r = -\frac{i}{2} (S - iT) \mathcal{Y}_i^j .
\end{aligned}$$

These equations yield

$$\begin{aligned}
\frac{1}{2} (\Omega_{-}^j \delta_i^l - 2i \Gamma_{-i}{}^j{}^l) K_l^k &= -K_m^k (\hat{E}_{-}^j K_i^k + A_{-n}^{\bullet+m} \hat{E}_{+}^n K_i^k) , \\
\frac{1}{2} (\Omega_{+i} \delta_k^j + 2i \Gamma_{+i k}{}^j) K_m^k &= -K_i^k (\hat{E}_{+k} K_m^j + A_{+k}^{-l} \hat{E}_{-l} K_m^j) ,
\end{aligned} \tag{5.15}$$

and

$$\{\hat{E}_{+k}, \hat{E}_{-}^m\} - A_{-n}^{\bullet+m} \{\hat{E}_{+k}, \hat{E}_{+}^n\} + A_{+k}^{-l} \{\hat{E}_{-l}, \hat{E}_{-}^m\} - A_{+k}^{-l} A_{-n}^{\bullet+m} \{\hat{E}_{-l}, \hat{E}_{+}^n\} = 0 , \tag{5.16}$$

and

$$\begin{aligned}
\hat{E}_{+k} A_{-n}^{\bullet+m} + A_{+k}^{-l} \hat{E}_{-l} A_{-n}^{\bullet+m} &= 0 , \\
\hat{E}_{-}^m A_{+k}^{-l} - A_{-n}^{\bullet+m} \hat{E}_{+}^n A_{+k}^{-l} &= 0 .
\end{aligned} \tag{5.17}$$

The next constraint $\{\nabla_{+i}, \nabla_{+}^j\} = i \delta_i^j \nabla_{\#}$ is equivalent to

$$\begin{aligned}
& \{E_{+i}, E_{+}^j\} + \frac{1}{2} \Omega_{+i} E_{+}^j + \frac{1}{2} \Omega_{+}^j E_{+i} + \\
& + i \Gamma_{+i k}{}^l \left(-\delta_l^j E_{+}^k + \frac{1}{2} \delta_l^k E_{-}^j \right) - i \Gamma_{+k}{}^j{}^l \left(\delta_i^k E_{+l} - \frac{1}{2} \delta_l^k E_{+i} \right) = i \delta_i^j \nabla_{\#} , \\
& E_{+i} \Omega_{+}^j + E_{+}^j \Omega_{+i} - \\
& -i \Gamma_{+i k}{}^l \left(\delta_l^j \Omega_{-}^k - \frac{1}{2} \delta_l^k \Omega_{+}^j \right) - i \Gamma_{+k}{}^j{}^l \left(\delta_i^k \Omega_{+l} - \frac{1}{2} \delta_l^k \Omega_{+i} \right) = 0 , \\
& -i E_{+i} \Gamma_{+}^j \mathcal{Y} + i E_{+}^j \Gamma_{+i} \mathcal{Y} - \frac{i}{2} \Omega_{+i} \Gamma_{+}^j \mathcal{Y} + \frac{i}{2} \Omega_{+}^j \Gamma_{+i} \mathcal{Y} - \\
& - \Gamma_{+i k}{}^l \left(\delta_l^j \Gamma_{+}^k \mathcal{Y} - \frac{1}{2} \delta_l^k \Gamma_{+}^j \mathcal{Y} \right) + \Gamma_{+k}{}^j{}^l \left(\delta_i^k \Gamma_{+l} \mathcal{Y} - \frac{1}{2} \delta_l^k \Gamma_{+i} \mathcal{Y} \right) + \\
& + \Gamma_{+i r}{}^l \Gamma_{+l}^j{}^s \mathcal{Y}_s^r + \Gamma_{+r}^j{}^l \Gamma_{+i l}{}^s \mathcal{Y}_s^r = 0 .
\end{aligned}$$

In particular, when $i = j$, the equations above determine $\nabla_{\#}$. When $i \neq j$, we find the additional constraints:

$$\{\hat{E}_{+k}, \hat{E}_{+}^m\} - A_{+n}^{\bullet+m} \{\hat{E}_{+k}, \hat{E}_{-}^n\} + A_{+k}^{-l} \{\hat{E}_{-l}, \hat{E}_{+}^m\} - A_{+k}^{-l} A_{+n}^{\bullet+m} \{\hat{E}_{-l}, \hat{E}_{-}^n\} = 0 , \tag{5.18}$$

and

$$\begin{aligned}\hat{E}_{+k} A_{+n}^{\bullet m-} + A_{+k}^{-l} \hat{E}_{-l} A_{+n}^{\bullet m-} &= 0, \\ \hat{E}_{+}^{\bullet m} A_{+k}^{-l} - A_{+}^{\bullet m-} \hat{E}_{+n}^{\bullet} A_{+k}^{-l} &= 0.\end{aligned}\tag{5.19}$$

Finally, from the last constraint in eq. (2.8), $\{\nabla_{-i}, \nabla_{-}^j\} = i\delta_i^j \nabla_{-}$, we find

$$\begin{aligned}& \{E_{-i}, E_{-}^j\} - \frac{1}{2}\Omega_{-i} E_{-}^j - \frac{1}{2}\Omega_{-}^j E_{-i} + \\ & + i\Gamma_{-i k}^l \left(-\delta_l^j E_{-}^k + \frac{1}{2}\delta_l^k E_{-}^j \right) - i\Gamma_{-k}^j{}^l \left(\delta_i^k E_{-l} - \frac{1}{2}\delta_l^k E_{-i} \right) = i\delta_i^j \nabla_{-}, \\ & E_{-i} \Omega_{-}^j + E_{-}^j \Omega_{-i} - \\ & - i\Gamma_{-i k}^l \left(\delta_l^j \Omega_{-}^k - \frac{1}{2}\delta_l^k \Omega_{-}^j \right) - i\Gamma_{-k}^j{}^l \left(\delta_i^k \Omega_{-l} - \frac{1}{2}\delta_l^k \Omega_{-i} \right) = 0, \\ & -iE_{-i} \Gamma_{-}^j \mathcal{Y} + iE_{-}^j \Gamma_{-i} \mathcal{Y} + \frac{i}{2}\Omega_{-i} \Gamma_{-}^j \mathcal{Y} - \frac{i}{2}\Omega_{-}^j \Gamma_{-i} \mathcal{Y} + \\ & + \Gamma_{-i k}^l \left(-\delta_l^j \Gamma_{+}^k \mathcal{Y} + \frac{1}{2}\delta_l^k \Gamma_{+}^j \mathcal{Y} \right) + \Gamma_{-k}^j{}^l \left(\delta_i^k \Gamma_{+l} \mathcal{Y} - \frac{1}{2}\delta_l^k \Gamma_{+i} \mathcal{Y} \right) + \\ & + \Gamma_{-i r}^l \Gamma_{-l}^j{}^s \mathcal{Y}_s^r + \Gamma_{-r}^j{}^l \Gamma_{-i l}^s \mathcal{Y}_s^r = 0.\end{aligned}$$

When $i = j$, it determines ∇_{-} . When $i \neq j$, some additional constraints on A 's arise, namely,

$$\{\hat{E}_{-k}, \hat{E}_{-}^m\} - A_{-n}^{\bullet m+} \{\hat{E}_{-k}, \hat{E}_{+}^n\} - A_{-k}^{+l} A_{-n}^{\bullet m+} \{\hat{E}_{+l}, \hat{E}_{+}^n\} + A_{-k}^{+l} \{\hat{E}_{+l}, \hat{E}_{+}^n\} = 0,\tag{5.20}$$

and

$$\begin{aligned}\hat{E}_{-k} A_{-n}^{\bullet m+} + A_{-k}^{+l} \hat{E}_{+l} A_{-n}^{\bullet m+} &= 0, \\ \hat{E}_{-}^m A_{-k}^{+l} - A_{-}^{\bullet m+} \hat{E}_{+n}^{\bullet} A_{-k}^{+l} &= 0.\end{aligned}\tag{5.21}$$

Putting all together allows us to determine the connections A_{+i}^{-j} and A_{-i}^{+j} . Eq. (5.16) actually breaks up into the two equations,

$$\begin{aligned}\hat{G}_{+k-}^{\bullet m\neq} - iA_{-k}^{\bullet +m} - A_{+k}^{-l} A_{-n}^{\bullet m+} \hat{G}_{-l+}^{\bullet n\neq} &= 0, \\ \hat{G}_{+k-}^{\bullet m=} + iA_{+k}^{-m} - A_{+k}^{-l} A_{-n}^{\bullet m+} \hat{G}_{-l+}^{\bullet n=} &= 0.\end{aligned}\tag{5.22}$$

Similarly, eq. (5.18) implies

$$\begin{aligned}i\delta_k^m + A_{+k}^{-l} \hat{G}_{-l+}^{\bullet m\neq} - A_{+}^{\bullet m-} \hat{G}_{+k-}^{\bullet n\neq} &= 0, \\ -iA_{+k}^{-l} A_{+l}^{\bullet m-} + A_{+k}^{-l} \hat{G}_{-l+}^{\bullet m=} - A_{+}^{\bullet m-} \hat{G}_{+k-}^{\bullet n=} &= 0.\end{aligned}\tag{5.23}$$

Eq. (5.20) also delivers the two equations as follows:

$$\begin{aligned} i\delta_k^m + A_{-k}^{+l} \hat{G}_{+l-}^{\bullet m=} - A_{-}^{\bullet m+} \hat{G}_{n-k+}^{\bullet n=} &= 0, \\ -iA_{-k}^{+l} A_{-}^{\bullet m+} + A_{-k}^{+l} \hat{G}_{+l-}^{\bullet m\neq} - A_{-}^{\bullet m+} \hat{G}_{n-k+}^{\bullet n\neq} &= 0. \end{aligned} \quad (5.24)$$

The first lines of eqs. (5.23) and (5.24) are quite remarkable, since they give the inhomogeneous *first-order* relations between the ‘undotted’ and ‘dotted’ components of the $SU(2)$ -tensor vector superfield A .

The anticommutator $\{\nabla_{+i}^{\bullet}, \nabla_{-}^j\}$ implies the second-order relations

$$\begin{aligned} \hat{G}_{-m+}^{\bullet k\neq} + iA_{-m}^{+k} - A_{+}^{\bullet k-} A_{-m}^{+n} \hat{G}_{+n-}^{\bullet l\neq} &= 0, \\ \hat{G}_{-m+}^{\bullet k=} - iA_{+}^{\bullet k-} A_{-m}^{+n} \hat{G}_{+n-}^{\bullet l=} &= 0. \end{aligned} \quad (5.25)$$

Eqs. (5.22) and (5.25), and eqs. (5.23) and (5.24) as well, are related by

$$\begin{aligned} (A_{-}^{+})_i^j &= (A_{+}^{-} \hat{G}_{-+}^{\neq})_i^l ((\hat{G}_{+-}^{\bullet =})^{-1})_l^j, \\ (A_{+}^{\bullet -})_i^j &= (A_{-}^{\bullet +} \hat{G}_{-+}^{\neq})_i^k ((\hat{G}_{+-}^{\bullet \neq})^{-1})_k^j, \end{aligned} \quad (5.26)$$

respectively. Altogether, they allow us to explicitly determine the superfield A . In the matrix notation it takes the simple form,

$$A_{+i}^{-j} = \left(\sqrt{(\hat{G}_{+-}^{\bullet =})(\hat{G}_{-+}^{\bullet \neq})^{-1}} \right)_i^j, \quad A_{-i}^{+j} = \left(\sqrt{(\hat{G}_{-+}^{\bullet \neq})(\hat{G}_{+-}^{\bullet =})^{-1}} \right)_i^j, \quad (5.27a)$$

and

$$A_{+i}^{\bullet -j} = \left(\sqrt{(\hat{G}_{+-}^{\bullet =})(\hat{G}_{-+}^{\bullet \neq})^{-1}} \right)_i^j, \quad A_{-i}^{\bullet +j} = \left(\sqrt{(\hat{G}_{-+}^{\bullet \neq})(\hat{G}_{+-}^{\bullet =})^{-1}} \right)_i^j. \quad (5.27b)$$

We conclude that substituting our Ansatz into the defining constraints (2.8) leads to both differential *and* algebraic equations on the superfields A ’s. The algebraic constraints fully determine that superfields and, hence, fix our Ansatz completely. The remaining differential equations (5.7), (5.9), (5.17), (5.19) and (5.21) ⁴ become constraints on the only remaining (4,4) superfield H^m . These constraints should eliminate the redundant irreducible (4,4) superfields in the general and reducible (4,4) superfield H^m , and leave only that (4,4) irreducible superfield which describes the off-shell (4,4) conformal supergravity multiplet. It is presently unclear to us how to find an explicit solution to the remaining highly complicated non-linear differential equations on the superfield H^m in terms of proper (4,4) superfield prepotentials, beyond the linearised solution [8, 9, 18] and a perturbation theory.

⁴They are not all independent, but they seem to be non-trivial, unlike that in the (2,2) case.

6 On the matter couplings in (4,4) supergravity

To describe the most general matter couplings in 2d, (4,4) supergravity, one needs to describe first all inequivalent (4,4) matter representations in two dimensions. Constructing the most general hypermultiplet couplings in (4,4) supergravity remains an unsolved problem, and we are not going to solve it here. Instead, we want to concentrate on the (4,4) supersymmetric matter to be represented by TM-I or TM-II whose selfinteractions and couplings to the (4,4) supergravity can be rather easily constructed in superspace, like that in four dimensions [30, 31, 32].⁵

The 2d, manifestly locally (4,4) supersymmetric action, which is quadratic in the TM-I matter superfields, can be written down in terms of the TM-I prepotentials (V, V_i^j) as follows [14]:

$$I = \int d^2z d^8\theta E^{-1} [VA + V_i^j A_j^i] , \quad (6.1)$$

where the full supervielbein superdeterminant E^{-1} and the TM-I superfields, A and $A_j^i = (\sigma_I)_j^i A_I$, whose leading components are the TM-I auxiliary fields (see sect. 3), have been introduced. The action (6.1) is invariant under the following gauge transformations of the prepotentials [34]:

$$\begin{aligned} \delta V &= \nabla^{i\alpha} \Lambda_{i\alpha} + \text{h.c.} , \\ \delta V_i^j &= \frac{i}{3} (\gamma_3)_\alpha{}^\beta \nabla_\beta^k \left[\delta_k^j \Lambda_i^\alpha - \frac{1}{2} \delta_i^j \Lambda_k^\alpha + \mathcal{C}^{jl} \Lambda_{(ikl)}^\alpha \right] . \end{aligned} \quad (6.2)$$

The action (6.1) can be rewritten in the chiral superspace as

$$I = \frac{1}{2} \int d^2z d^4\theta \mathcal{E}^{-1} \Phi^2 + \text{h.c.} , \quad (6.3)$$

where the chiral superspace density \mathcal{E} and the reduced covariantly chiral (4,4) superfield Φ (see appendix C) have been introduced. Eq. (6.3) can be further generalised to

$$I_V = \frac{1}{2} \int d^2z d^4\theta \mathcal{E}^{-1} V(\Phi) + \text{h.c.} , \quad (6.4)$$

where $V(\Phi)$ is a *homogeneous* function of degree *two*, while maintaining all of the (4,4) superconformal symmetries. Eq. (6.4) is quite similar to the standard couplings of the N=2 vector multiplets to the N=2 supergravity in *four* dimensions [31]. However, there are also some important differences which originate from dimensional reduction (see also Appendix C).

⁵See ref. [33] for a recent review.

The geometrical meaning of the action (6.4) can be most easily understood after rewriting it in terms of the conventional (2,2) superfields in two dimensions. The resulting action appears to be the special case of the general N=2 supersymmetric *non-linear sigma-model* (NLSM) coupled to the N=2 supergravity, and described by the action

$$I_{N=2} = \int d^2x \left\{ \int d^2\theta d^2\bar{\theta} E^{-1} K(\phi, \bar{\phi}) - \int d^2\theta \mathcal{E}^{-1} W(\phi) - \int d^2\theta \mathcal{E}^{-1} R\Upsilon(\phi) + \text{h.c.} \right\} , \quad (6.5)$$

in terms of the (2,2) Kähler potential K , the superpotential W and the dilaton field Υ (all are functions of (2,2) chiral superfields ϕ^a representing (2,2) matter), where \mathcal{E} is the (2,2) chiral density and R is the (2,2) chiral superfield strength of the (2,2) supergravity which was already introduced in eq. (4.2). In the (2,2) case, all the functions K , W and Υ are independent off-shell, while the W and Υ are holomorphic. On-shell, after eliminating the N=2 matter auxiliary fields via their algebraic equations of motion, that functions turn out to be related as [35, 36]

$$W = \left[\frac{\partial^2 K}{\partial \phi^a \partial \bar{\phi}^b} \right]^{-1} \frac{\partial \Upsilon^*}{\partial \bar{\phi}^b} \left(\frac{\partial W}{\partial \phi^a} + H \frac{\partial \Upsilon}{\partial \phi^a} \right) , \quad (6.6)$$

where the non-propagating complex auxiliary field $H = R|$ of the (2,2) supergravity multiplet has been introduced. In the particular case of a *single* chiral superfield ϕ , it is always possible to make the dilaton field linear, $\Upsilon = \phi$, by field redefinition. Then eq. (6.6) forces the Kähler metric to be flat, $K = \bar{\phi}\phi$, and gives rise to the *Liouville* potential, $W(\phi) = \mu e^\phi + H$ [35, 36].

Being rewritten in the N=2 superspace to eq. (6.5), the (4,4) supersymmetric action (6.4) determines all the functions K , W and Υ in terms of the only holomorphic (and homogeneous of degree two) function V . One may wonder about the appearance of the potential and the Fradkin-Tseytlin-type term in the action (6.5) resulting from the action (6.4), because these terms seem to be inconsistent with the classical N=4 superconformal invariance of the theory under consideration. It is nevertheless possible to have both such terms in the superconformally-invariant action if they appear as the result of *spontaneous* supersymmetry breaking, which triggers the spontaneous conformal symmetry breaking as well. Needless to say, it always leads to very special potentials generalising the Liouville one.

To give a simple example, let us temporarily switch off the (4,4) supergravity fields in the (4,4) supersymmetric action (6.4). By using the results of Appendix C and eliminating the (4,4) matter auxiliary fields via their algebraic equations of motion, one arrives at the following bosonic part of the lagrangian describing the

purely matter part of eq. (6.4) (*cf.* ref. [37]):

$$\begin{aligned}
L_B = & \frac{\partial^2 H}{\partial N^a \partial N^b} \left\{ \partial_\mu N^a \partial^\mu N^b + \partial_\mu M^a \partial^\mu M^b + \partial_\mu Q^a \partial^\mu Q^b + \partial_\mu P^a \partial^\mu P^b \right\} \\
& + 2 \frac{\partial^2 H}{\partial N^a \partial M^b} \varepsilon^{\mu\nu} \partial_\mu Q^a \partial_\nu P^b - 2 \frac{\partial^2 H}{\partial N^a \partial N^b} m^a m^b \\
& - 2 m^a \frac{\partial^2 H}{\partial M^a \partial N^b} \left(\left[\frac{\partial^2 H}{\partial N \partial N} \right]^{-1} \right)^{bc} \frac{\partial^2 H}{\partial M^c \partial N^d} m^d,
\end{aligned} \tag{6.7}$$

where we have used the notation (see Appendix C)

$$A = \frac{1}{\sqrt{2}}(M + iN), \quad B = \frac{1}{\sqrt{2}}(P + iQ), \quad \text{and} \quad H(M, N) = \text{Im } V(A). \tag{6.8}$$

The dimensionful constants m^a , which appear in eq. (6.7), arise in the process of dimensional reduction as the expectation values of some auxiliary fields. It now becomes clear that we are dealing with the NLSM having the torsion and the potential induced by the spontaneous supersymmetry breaking ($m^a \neq 0$) via dimensional reduction. It should also be noticed that, due to the torsion alone, the NLSM target space geometry in eq. (6.7) is *not* quaternionic, which agrees with the general results of de Wit and van Nieuwenhuizen [38]. By analogy with the NLSM counterpart in four dimensions, describing the scalar kinetic terms resulting from a chiral integral of a holomorphic function of N=2 (abelian) reduced chiral superfields [31], we call the NLSM target space geometry of eq. (6.7) *special*. Eq. (6.7) reduces to the free form if the function V is quadratic in the fields.

Once the action (6.4) is known to have a superpotential, it must also possess the non-vanishing Fradkin-Tseytlin-type term because of eq. (6.6). The action (6.4) may be suitable for describing the *non-critical* (4,4) strings propagating in the background spaces having special geometry [36]. It should be noticed here that quantum-mechanically consistent *critical* N=4 strings do not exist, even in case of a non-trivial background space [39], but there is no problem with constructing quantum-mechanically consistent models of non-critical N=4 strings [24]. Remarkably, no additional restrictions on the (4,4) supersymmetric NLSM geometry arise from the NLSM quantum perturbation theory, since any (4,4) supersymmetric NLSM has no UV divergences at all.⁶

As far as the TM-II matter theories in the curved superspace of (4,4) supergravity are concerned, they are extremely restricted and, until recently, no such examples were

⁶As far as the NLSM of eq. (6.7) is concerned, its UV finiteness was explicitly proved in ref. [37] by using the (4,4) superfield perturbation theory in two dimensions.

constructed. The TM-II auxiliary fields can be considered as the leading components of a TM-I defining the so-called *kinetic* (4,4) multiplet, like that in 4d. Therefore, TM-I and TM-II are *dual* to each other, though they are not equivalent [25]. There exists the locally (4,4) supersymmetric invariant given by a product of TM-I and TM-II [21, 22, 23, 25]. In the curved (4,4) superspace, this invariant takes the form [36]

$$I_{\text{I-II}} = \int d^2x d^4\theta d^4\bar{\theta} E^{-1} (\Pi S + \Xi T) + \left[\int d^2x d^4\theta \mathcal{E}^{-1} \Lambda R + \text{h.c.} \right] , \quad (6.9)$$

where the real superfield prepotentials Π and Ξ , and the chiral superfield prepotential Λ , of the TM-II have been introduced [25].

The rigidly (4,4) supersymmetric invariant describing the free TM-II action, which is quadratic in the fields, is known [25]. However, its locally (4,4) supersymmetric generalisation does not exist.⁷ When being compared to the rigid (4,4) supersymmetry, the allowed matter couplings in the (4,4) conformal supergravity are much more restricted, and it is also known to be the case for the N=2 matter couplings in the four-dimensional N=2 supergravity [31]. As far as the TM-II in 2d is concerned, this problem is only apparent, since there exists its *improved* (i.e. superconformally invariant) 2d action [36], which can be coupled to the 2d, (4,4) conformal supergravity. The point is that it is possible to form the TM-I out of the TM-II components in yet another *non-linear* way, namely,

$$\begin{aligned} R_{\text{impr.}} &= L^{-1} U^* + \frac{4i}{3} \chi_{+i}^{\bullet} \chi_{-}^{\bullet j} L_j^i L^{-3} , \\ S_{\text{impr.}} &= \frac{1}{4} L^{-1} M + \frac{2i}{3} \left(\chi_{+i}^{\bullet} \chi_{-}^j - \chi_{-}^{\bullet j} \chi_{+i} \right) L_j^i L^{-3} , \\ T_{\text{impr.}} &= \frac{1}{4} L^{-1} N + \frac{2}{3} \left(\lambda_{+i}^{\bullet} \chi_{-}^j + \chi_{-}^{\bullet j} \lambda_{+i} \right) L_j^i L^{-3} , \end{aligned} \quad (6.10)$$

where we have used the notation

$$L \equiv \sqrt{L_i^j (L_i^j)^*} . \quad (6.11)$$

As was shown in sect. 3, the (4,4) superspace constraints (2.8) have the hidden super-Weyl symmetry, the (4,4) supergravity field strengths are represented by TM-I, and TM-II appears as a scale compensator. The non-linear realisation of TM-I can be derived from a calculation of the *finite* form of the super-Weyl transformations. One finds eq. (6.10) either in the lowest order of an expansion of the finite super-Weyl transformation in powers of the TM-I fields, or, equivalently, in a superconformally

⁷The same is true in four dimensions [30].

flat gauge where the (untransformed) TM-I fields are set to zero. The infinitesimal super-Weyl transformations given in eq. (3.6) vanish in the superconformally flat gauge.

Eq. (6.9) can now be used to define an invariant coupling of the improved TM-II to the (4,4) supergravity in the form

$$I_{\text{impr.}} = \int d^2x d^4\theta d^4\bar{\theta} E^{-1} (\Pi S_{\text{impr.}} + \Xi T_{\text{impr.}}) + \left[\int d^2x d^4\theta \mathcal{E}^{-1} \Lambda R_{\text{impr.}} + \text{h.c.} \right] . \quad (6.12)$$

The existence of the improved TM-II in 2d is a direct consequence of the existence of the improved N=2 tensor multiplet in 4d [30], since they are related via dimensional reduction. Unlike the improved N=2 tensor multiplet in 4d, its 2d counterpart does not have any gauge degrees of freedom, which allows the (4,4) locally supersymmetric component action associated with eq. (6.12) to have the manifest $SU(2)$ internal symmetry.

There actually exists an additional resource to build yet another (4,4) locally supersymmetric invariant, namely, the so-called *Fayet-Iliopoulos* (FI) term.⁸ In the curved (4,4) superspace, the FI term takes the form [36]

$$I_{\text{FI}} = -2\mu \int d^2x d^4\theta d^4\bar{\theta} E^{-1} \Pi . \quad (6.13)$$

Given the action

$$I_{\text{L}} = I_{\text{impr.}} + I_{\text{FI}} , \quad (6.14)$$

the auxiliary field M of the improved TM-II enters this action in the combination $e^{-\phi} M^2 - 2\mu M$. Eliminating this auxiliary field via its algebraic equation of motion gives rise to the *Liouville* potential $-\mu^2 e^{\phi}$ again. The action (6.14) is therefore the (4,4) locally supersymmetric Liouville action.

7 Conclusion

In this paper we considered the superfield structure of the (4,4) conformal supergravity in two dimensions, and made progress in finding a solution to its superspace constraints. Even though the remaining problems are of technical nature, a deeper insight into the complicated superspace structure of the (4,4) supergravity theory may be needed in order to get the explicit solution. As a next task, a detailed comparison with the linearised analysis may be useful, in order to find the best way to proceed.

⁸The FI term was used in ref. [23] to construct the rigidly (4,4) supersymmetric Liouville action.

Another aspect deserving further investigations is the (4,4) locally supersymmetric model-building, i.e. constructing the matter couplings in 2d, (4,4) supergravity. We discussed in this paper only two (4,4) scalar multiplets, TM-I and TM-II, whereas different variant representations of hypermultiplet are also known to exist [25]. It would be of interest to describe them also. The (4,4) non-critical strings and the NLSM special geometry are the natural areas for possible applications of such models.

The very framework of the conventional (4,4) superspace used above may happen to be inadequate for describing the *most* general matter couplings in the (4,4) supergravity, so that the more powerful *harmonic* superspace method [7] may be needed. The $SU(2) \times SU(2)$ harmonic (4,4) superspace approach recently proposed by Ivanov and Sutulin [40, 41] may be the proper way to address general issues. The $SU(2) \times SU(2)$ harmonic (4,4) superspace has two independent sets of harmonic variables and the necessarily infinite sets of auxiliary fields for an off-shell hypermultiplet and an off-shell (4,4) supergravity, which make the transition from any harmonic superspace formulation to components highly non-trivial. The existing resources of the conventional (4,4) superspace deserve to be explored further, in parallel with the complementary harmonic superspace approach.

We summarize the component results about the 2d, (4,4) supergravity and the (4,4) string action in Appendix B. The list of symmetries of the (4,4) string action is also given in Appendix B. It includes both the known continuous symmetries and the new discrete symmetries of the string action.

Acknowledgements

One of the authors (S.V.K.) would like to thank Jim Gates, Marc Grisaru, Evgeni Ivanov, Jens Schnittger and Marcia Wehlau for many stimulating discussions.

Appendix A: notation and conventions

We use small greek letters λ, μ, ν, \dots for the vector indices associated with the two-dimensional curved spacetime or the string world-sheet, and small latin letters a, b, c, \dots for the vector indices in the corresponding tangent space. The two-dimensional Minkowski metric is given by

$$\eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \quad (A.1)$$

Given ϵ_i to represent a Dirac spinor in the fundamental representation of $SU(2)$, small latin letters i, j, k, \dots are used to denote the $SU(2)$ indices, $i = 1, 2$. The $SU(2)$ indices are ‘canonically’ contracted from the upper left to the lower right, and they are raised (lowered) with \mathcal{C}^{ij} (\mathcal{C}_{ij}), so that the following identities hold:

$$\epsilon^i \equiv \mathcal{C}^{ij} \epsilon_j , \quad \epsilon_i \equiv \epsilon^j \mathcal{C}_{ji} , \quad \mathcal{C}^{ik} \mathcal{C}_{kj} = -\delta^i_j . \quad (A.2)$$

Explicitly, we have

$$\mathcal{C}^{ij} = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (A.3)$$

The complex conjugation acts on the $SU(2)$ indices in the following way:

$$(\epsilon_i)^* \equiv \epsilon^{*i} . \quad (A.4)$$

Thus, we deduce that $\mathcal{C}_{ij} = (\mathcal{C}^{ij})^*$ and

$$(\epsilon^i)^* = (\mathcal{C}^{ik} \epsilon_k)^* = \mathcal{C}_{ik} \epsilon^{*k} = -\epsilon^{*i} . \quad (A.5)$$

The Majorana and Dirac conjugations of spinors are defined as follows:

$$\tilde{\epsilon}^i = (\epsilon^i)^T C , \quad \bar{\epsilon}^i = (\epsilon_i)^\dagger C , \quad (A.6)$$

where $C_{\alpha\beta} = \sigma^2$ is the charge conjugation matrix which obeys

$$C = C^\dagger , \quad C \gamma^a C^{-1} = -(\gamma^a)^T . \quad (A.7)$$

We find it useful to introduce the light-cone coordinates

$$x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^1) , \quad (A.8)$$

in terms of the coordinates $x^a = (x^0, x^1)$ of the tangent space. The index values $a = 0, 1$ here should not be confused with the similar values for the target space indices.

The light-cone components ϵ_{\pm} of a spinor ϵ define the one-dimensional representations of the Lorentz group $SO(1, 1)$. The field ϵ_+ (ϵ_-) moves to the right (left), and they have the following transformation properties under the action of the $SO(1, 1)$ generator γ_3 :

$$\gamma_3 \epsilon_- = -\epsilon_- , \quad \gamma_3 \epsilon_+ = \epsilon_+ . \quad (A.9)$$

We identify the spinor components of ϵ with its light-cone components ϵ_{\pm} ,

$$\epsilon_i = \begin{pmatrix} \epsilon_{+i} \\ \epsilon_{-i} \end{pmatrix} . \quad (A.10)$$

The two-dimensional gamma matrices satisfy the algebra

$$(\gamma^a)_{\alpha}{}^{\delta} (\gamma^b)_{\delta}{}^{\beta} = \eta^{ab} \delta_{\alpha}{}^{\beta} + \varepsilon^{ab} (\gamma_3)_{\alpha}{}^{\beta} , \quad (A.11)$$

where ε^{ab} is the Levi-Civita symbol, $\varepsilon^{01} = 1$. Explicitly, we choose $(\gamma_0)_{\alpha}{}^{\beta} = -i\sigma^2$, $(\gamma_1)_{\alpha}{}^{\beta} = \sigma^1$, and $(\gamma_3)_{\alpha}{}^{\beta} = (\gamma^0 \gamma^1)_{\alpha}{}^{\beta} = \sigma^3$, or, equivalently,

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (A.12)$$

In our component formulae to be given in Appendix B, all the spinor indices are omitted, as a rule. In addition, the following identities hold:

$$\begin{aligned} \{\gamma^a, \gamma^b\} &= 2\eta^{ab} , \\ [\gamma^a, \gamma^b] &= 2\varepsilon^{ab} \gamma_3 , \\ \varepsilon^{ab} \varepsilon_{cd} &= -(\delta_c^a \delta_d^b - \delta_d^a \delta_c^b) , \\ \gamma_3 \gamma_a &= \varepsilon_{ab} \gamma^b . \end{aligned} \quad (A.13)$$

As far as the curved 2d spacetime or the string world-sheet is concerned, we have the following relation for the gamma matrices:

$$\gamma^{\mu} \gamma^{\nu} = \eta^{\mu\nu} + e^{-1} \varepsilon^{\mu\nu} \gamma_3 , \quad (A.14)$$

where e is the determinant of the zweibein $e_{\mu}{}^a$, and $e^{-1} \varepsilon^{\mu\nu}$ is the Levi-Civita tensor density.

The Pauli matrices $(\sigma^I)_i{}^j$ satisfy the algebra

$$(\sigma^I \sigma^J)_i{}^j = \delta^{IJ} \delta_i{}^j + i\varepsilon^{IJK} (\sigma^K)_i{}^j , \quad (\sigma^I)_{ij} \equiv (\sigma^I)_i{}^k \mathcal{C}_{kj} = (\sigma^I)_{ji} , \quad (A.15)$$

and have the usual form

$$(\sigma^1)_i{}^j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2)_i{}^j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\sigma^3)_i{}^j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.16})$$

A calculation of the spinor bilinear relations in N=4 supersymmetry is similar to that in N=1 or N=2 supersymmetry, but it also has some additional features due to the $SU(2)$ structure. For instance, we find that

$$\begin{aligned} \tilde{\epsilon}^i \psi_i &= (\epsilon^i)^T C \psi_i = (\mathcal{C}^{ij} \epsilon_j)^T C \psi_i = (\mathcal{C}^{ij})^T \epsilon_j^\alpha C_{\alpha\beta} \psi_i^\beta = \psi_i^\beta C_{\beta\alpha} (\mathcal{C}^{ij})^T \epsilon^{j\alpha} \\ &= (\mathcal{C}^{ij} \psi_i)^T C \epsilon_j = -(\psi^i)^T C \epsilon_i = -\tilde{\psi}^i \epsilon_i. \end{aligned} \quad (\text{A.17})$$

Eq. (A.17) implies, in particular, that $\tilde{\psi}^i \psi_i = 0$. As far as the spinor bilinears with Pauli matrices are concerned, we find

$$\begin{aligned} \tilde{\epsilon}^i (\sigma^I)_i{}^j \psi_j &= (\mathcal{C}^{ik} \epsilon_k)^T C (\sigma^I)_i{}^j \psi_j = (\epsilon_k)^T C \mathcal{C}^{ki} (\sigma^I)_i{}^j \psi_j = \epsilon_k^\alpha C_{\alpha\beta} (\sigma^I)^{kj} \psi_j^\beta \\ &= \psi_j^\beta C_{\beta\alpha} (\sigma^I)^{jk} \epsilon_k^\alpha = (\psi_j)^T C \mathcal{C}^{ji} (\sigma^I)_i{}^k \epsilon_k = (\mathcal{C}^{ij} \psi_j)^T C (\sigma^I)_i{}^k \epsilon_k \\ &= \tilde{\psi}^i (\sigma^I)_i{}^k \epsilon_k. \end{aligned} \quad (\text{A.18})$$

Because of the relation $C \gamma^a C^{-1} = -(\gamma^a)^T$, the contraction of spinor indices over a gamma matrix yields

$$\tilde{\epsilon}^i \gamma_a \psi_i = \tilde{\psi}^i \gamma_a \epsilon_i. \quad (\text{A.19})$$

Appendix B: (4,4) supergravity in components

An off-shell multiplet $(e_\mu{}^a, \psi_{\mu i}, A_\mu{}^I, R, S, T)$ of the 2d, minimal $N = 4$ conformal supergravity was given in refs. [14, 15] (see also refs. [42, 43] for the earlier results). We have $e_\mu{}^a$ for the zweibein, a complex Dirac spinor $\psi_{\mu i}$ in the $SU(2)$ doublet-representation for the gravitini, and $A_\mu{}^I$ as the real $SU(2)$ gauge field in the triplet-representation. The scalars S, T and R are all the auxiliary fields. The fields S and T are real, whereas the field R is complex. Altogether, this gives $(8+8)$ bosonic and fermionic degrees of freedom off-shell.

The infinitesimal transformation laws for the 2d, N=4 conformal supergravity

fields read (*cf.* refs. [14, 15])

$$\begin{aligned}
\delta_Q e_\mu^a &= -\frac{1}{2}\bar{\epsilon}^i \gamma^a \psi_{\mu i} + \frac{1}{2}\bar{\psi}_\mu^i \gamma^a \epsilon_i , \\
\delta_Q \psi_{\mu i} &= D_\mu \epsilon_i + \gamma_\mu \left[\frac{1}{4}(S - i\gamma_3 T) \epsilon_i + \frac{i}{2} R \gamma_3 \epsilon_i^* \right] , \\
\delta_Q A_\mu^I &= \frac{i}{4}\bar{\epsilon}^i (\sigma^I)_i^j \gamma_\mu \gamma_3 \varepsilon^{\rho\sigma} D_\rho \psi_{\sigma j} - \frac{i}{4}\bar{\epsilon}^i (\sigma^I)_i^j \gamma_\mu \gamma^\rho \left[\frac{1}{4}(S - i\gamma_3 T) \psi_{\rho j} + \frac{i}{2} R \gamma_3 \psi_{\rho j}^* \right] \\
&\quad - \frac{i}{4}\bar{\psi}_\nu^i \gamma_\mu \gamma^\nu (\sigma^I)_i^j \left[\frac{1}{4}(S - i\gamma_3 T) \epsilon_j + \frac{i}{2} R \gamma_3 \epsilon_j^* \right] + \text{h.c.} , \\
\delta_Q S &= -\varepsilon^{\mu\nu} \bar{\epsilon}^i \gamma_3 D_\mu \psi_{\nu i} + \bar{\epsilon}^i \gamma^\mu \left[\frac{1}{4}(S - i\gamma_3 T) \psi_{\mu i} + \frac{i}{2} R \gamma_3 \psi_{\mu i}^* \right] + \text{h.c.} , \\
\delta_Q T &= i\varepsilon^{\mu\nu} \bar{\epsilon}^i D_\mu \psi_{\nu i} - i\bar{\epsilon}^i \gamma_3 \gamma^\mu \left[\frac{1}{4}(S - i\gamma_3 T) \psi_{\mu i} + \frac{i}{2} R \gamma_3 \psi_{\mu i}^* \right] + \text{h.c.} , \\
\delta_Q R &= i\varepsilon^{\mu\nu} \bar{\epsilon}^i D_\mu \psi_{\nu i} - i\bar{\epsilon}^i \gamma_3 \gamma^\mu \left[\frac{1}{4}(S - i\gamma_3 T) \psi_{\mu i} + \frac{i}{2} R \gamma_3 \psi_{\mu i}^* \right] , \\
\delta_Q R^* &= -i\varepsilon^{\mu\nu} \bar{\epsilon}^i D_\mu \psi_{\nu i}^* + i\bar{\epsilon}^i \gamma_3 \gamma^\mu \left[\frac{1}{4}(S + i\gamma_3 T) \psi_{\mu i}^* + \frac{i}{2} R^* \gamma_3 \psi_{\mu i} \right] ,
\end{aligned} \tag{B.1}$$

where we have introduced D_μ as the covariant derivative,

$$D_\mu \epsilon_i = \partial_\mu \epsilon_i + \frac{1}{4} \omega_\mu^{ab} (e, \psi) \varepsilon_{ab} \gamma_3 \epsilon_i - i A_\mu^I (\sigma_I)_i^j \epsilon_j . \tag{B.2}$$

The spin connection $\omega_\mu^{ab}(e, \psi)$ is given by

$$\begin{aligned}
\omega_\mu^{ab}(e, \psi) &= \omega_\mu^{ab}(e) - \frac{1}{4} (\bar{\psi}_\mu^i \gamma^a \psi_{\mu i}^b - \bar{\psi}_\mu^i \gamma^b \psi_{\mu i}^a \\
&\quad + \bar{\psi}^{ai} \gamma_\mu \psi_{\mu i}^b - \bar{\psi}^{bi} \gamma^a \psi_{\mu i} + \bar{\psi}^{ai} \gamma^b \psi_{\mu i} - \bar{\psi}^{bi} \gamma_\mu \psi_{\mu i}^a) ,
\end{aligned} \tag{B.3}$$

where $\omega_\mu^{ab}(e)$ is the usual (torsion-free) spin connection,

$$\begin{aligned}
\omega_\mu^{ab}(e) &= \frac{1}{2} e^{\nu a} (\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) - \frac{1}{2} e^{\nu b} (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) \\
&\quad - \frac{1}{2} e^{\rho a} e^{\lambda b} (\partial_\rho e_{\lambda c} - \partial_\lambda e_{\rho c}) e_\mu^c .
\end{aligned} \tag{B.4}$$

Appendix C: $N = 2$ real chiral superfield in four dimensions, and its dimensional reduction to $d = 2$

It is often useful to formulate field theories with extended supersymmetry in higher dimensions and then dimensionally reduce them to lower dimensions. As far as the 2d, $N = 4$ field theories are concerned, one can use either the $N = 1$ superfields in six dimensions or the $N = 2$ superfields in four dimensions (4d). Taking the latter choice, the simplest $N = 2$ superspace constraints defining an $N = 2$ chiral scalar superfield $\Phi(x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i})$, $\mu = 0, 1, 2, 3$ and $i, j, \dots = 1, 2$, are given by ⁹

$$D_\alpha^i \bar{\Phi} = \bar{D}_{\dot{\alpha}i} \Phi = 0 , \tag{C.1}$$

⁹ In this Appendix C we use the standard four-dimensional notation [2, 18].

where we have introduced the (flat) superspace covariant derivatives D_α^i and $\overline{D}_{\alpha i}^\bullet$ satisfying the algebra

$$\left\{ D_\alpha^i, \overline{D}_{j\beta}^\bullet \right\} = 2i\delta_j^i \sigma_{\alpha\beta}^\mu \partial_\mu . \quad (C.2)$$

The complex supermultiplet $(\Phi, \overline{\Phi})$ is reducible to the (generalised) real one by using the additional constraint [44]

$$\frac{1}{12} D_i^\alpha D_{\alpha j} D^{\beta i} D_\beta^j \Phi = \square \overline{\Phi} . \quad (C.3)$$

The solution to the constraints (C.2) and (C.3) reads ¹⁰

$$\begin{aligned} \Phi = & \exp \left\{ -\frac{i}{2} \theta_i^\gamma \tilde{\sigma}_{\gamma\gamma}^\mu \partial_\mu \bar{\theta}^{\dot{\gamma} i} \right\} \left[A + \theta_i^\alpha \psi_\alpha^i - \frac{1}{2} \theta_i^\alpha (\sigma^I)^i_j \theta_\alpha^j C^I \right. \\ & \left. + \frac{1}{8} \theta_i^\alpha (\sigma_{\mu\nu})_\alpha^\beta \theta_\beta^i F^{\mu\nu} - i(\theta^3)^{i\alpha} \tilde{\sigma}_{\alpha\beta}^\mu \partial_\mu \bar{\psi}_i^\beta + \theta^4 \square \overline{A} \right] , \end{aligned} \quad (C.4)$$

in terms of the components

$$(A, \psi_\alpha^i, C^I, F_{\mu\nu}) , \quad (C.5)$$

where A is a complex scalar, ψ^i is a 4d Majorana spinor isodoublet, C^I is a real isovector, and $F_{\mu\nu}$ is a real antisymmetric tensor satisfying the constraint

$$\partial^\mu F_{\mu\nu} = 0 . \quad (C.6)$$

Because of the constraint (C.6), the tensor $\tilde{F}_{\mu\nu}$ dual to $F_{\mu\nu}$ can be interpreted as the field strength of a vector [44, 45]. Accordingly, the supermultiplet (C.5) is usually referred to as the $N = 2$ *vector* multiplet in four dimensions.

The dimensional reduction to two dimensions amounts to $\partial_2 = \partial_3 = 0$. The 4d isospinor ψ^i can then be represented in terms of 2d spinors as

$$\psi^i = \begin{pmatrix} \psi^i \\ \tilde{\psi}^i \end{pmatrix} , \quad \tilde{\psi}^i \equiv C^{ij} C \bar{\psi}_j^T , \quad (C.7)$$

where C is the 2d charge conjugation matrix defined in Appendix A.

The constraint (C.6) can be easily solved *after* the dimensional reduction [37],

$$\begin{aligned} F_{01} &= m = \text{const} , & F_{23} &= D , \\ F_{\mu 2} &= \frac{1}{2} \varepsilon_{\mu\nu} \partial^\nu (B + \overline{B}) , & F_{\mu 3} &= \frac{1}{2i} \varepsilon_{\mu\nu} \partial^\nu (B - \overline{B}) , \end{aligned} \quad (C.8)$$

in terms of a complex scalar B and a real scalar D , where an arbitrary dimensionful constant m appears, in general. As a result, one arrives at the 2d, $N = 4$ twisted scalar multiplet,

$$(A, B, \psi^i, C^I, D) , \quad (C.9)$$

¹⁰We define $(\theta^3)^{i\alpha} = (\partial/\partial\theta_{i\alpha})\theta^4$ and $\theta^4 = \frac{1}{12} \theta_i^\alpha \theta_{\alpha j} \theta^{\beta i} \theta_\beta^j$.

comprising two complex scalars A and B , a 2d Dirac spinor isodoublet ψ^i and the auxiliary fields: a real isovector C^I and a real scalar D ($8_B \oplus 8_F$ components). It is called the TM-I, according to the classification proposed in ref. [25]. The transformation laws for the TM-I version of hypermultiplet components, which are obtained via dimensional reduction from the transformation laws of the 4d, N=2 vector multiplet are given by [37]

$$\begin{aligned}
\delta A &= \bar{\varepsilon}_i \frac{1}{2} (1 - \gamma_3) \psi^i + \bar{\psi}^i \frac{1}{2} (1 - \gamma_3) \varepsilon_i , \\
\delta B &= \bar{\varepsilon}_i \gamma_3 \tilde{\psi}^i , \\
\delta \psi^i &= (\sigma^I)^i_j C^I \gamma_3 \varepsilon^j - \frac{1}{2} (1 - \gamma_3) i \tilde{\partial} A \varepsilon^i + \frac{1}{2} (1 + \gamma_3) i \tilde{\partial} \bar{A} \varepsilon^i \\
&\quad - i D \varepsilon^i + 2m \gamma_3 \varepsilon^i + i \tilde{\partial} \varepsilon^i \bar{B} , \\
\delta C^I &= -\frac{1}{2} \bar{\varepsilon}_i \tilde{\partial} (\sigma^I)_j^i \psi^j + \text{h.c.} , \\
\delta D &= \frac{1}{2} \bar{\varepsilon}_i \gamma_3 \tilde{\partial} \psi^i + \text{h.c.} ,
\end{aligned} \tag{C.10}$$

where ε^i are the 2d, infinitesimal (4,4) supersymmetry parameters forming a Dirac spinor isodoublet, $\tilde{\partial} = \gamma^\mu \partial_\mu$, and γ^μ , $\mu = 0, 1$, are the 2d Dirac matrices defined in Appendix A.

The non-vanishing dimensionful constant m triggers a spontaneous breakdown of the (4,4) supersymmetry. It is already obvious from the (4,4) supersymmetry transformation law for the spinor fields ψ^i in eq. (C.10) whose right-hand side contains the Goldstone term (see sect. 6 also).

References

- [1] J. Wess and B. Zumino, Phys. Lett. **66B** (1977) 361.
- [2] J. Bagger and J. Wess, *Supersymmetry and Supergravity*, Princeton: Princeton University Press, 1983.
- [3] S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace or One Thousand and One Lessons in Supersymmetry*, Benjamin-Cummings Publ. Company Inc., Reading MA, 1983.
- [4] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity: A Walk through Superspace*, IOP Publishing Ltd., Bristol and Philadelphia, 1995.
- [5] D. Knizhnik, Usp. Fiz. Nauk **159** (1989) 401.
- [6] W. Siegel, Nucl. Phys. **B142** (1978) 301.
- [7] A. A. Galperin, E. A. Ivanov, S. Kalitzin, V. I. Ogievetsky and E. Sokatchev, Class. and Quantum Grav. **1** (1984) 469; *ibid.* **2** (1985) 601; *ibid.* **2** (1985) 617.
- [8] S. J. Gates Jr., and W. Siegel, Nucl. Phys. **B147** (1979) 77; *ibid.* **187** (1981) 389; *ibid.* **189** (1981) 295; *ibid.* **195** (1982) 39.
- [9] S. V. Ketov, Sov. Phys. Journ. **29** (1986) 416; *ibid.* **29** (1986) 800.
- [10] S. J. Gates Jr., M. T. Grisaru, M. Roček and P. K. Townsend, Nucl. Phys. **B286** (1987) 1.
- [11] S. J. Gates Jr., and H. Nishino, Class. and Quantum Grav. **3** (1986) 134.
- [12] M. Evans and B. A. Ovrut, Phys. Lett. **186B** (1987) 134.
- [13] A. Alnowaiser, Class. and Quantum Grav. **7** (1990) 1033.
- [14] S. J. Gates Jr., L. Lu and R. Oerter, Phys. Lett. **218B** (1989) 33.
- [15] S. J. Gates Jr., S. J. Hassoun und P. van Nieuwenhuizen, Nucl. Phys. **B317** (1989) 302.
- [16] M. T. Grisaru and M. E. Wehlau, Int. Journ. Mod. Phys. **A10** (1995) 753.
- [17] M. T. Grisaru and M. E. Wehlau, Nucl. Phys. **B457** (1995) 219.

- [18] S. V. Ketov, Fortschr. Phys. **36** (1988) 361.
- [19] M. Müller, Nucl. Phys. **B289** (1987) 557.
- [20] S. V. Ketov and I. V. Tyutin, Sov. J. Nucl. Phys. **44** (1986) 169.
- [21] E. A. Ivanov and S. O. Krivonos, J. Phys. A: Math. Gen. **17** (1984) L671.
- [22] E. A. Ivanov and S. O. Krivonos and V. M. Leviant, Nucl. Phys. **B304** (1988) 601.
- [23] O. Gorovoy and E. A. Ivanov, Nucl. Phys. **B381** (1992) 394.
- [24] C. Kounnas, M. Porrati and B. Rostand, Phys. Lett. **258B** (1991) 61.
- [25] S. J. Gates, Jr., and S. V. Ketov, *2D (4,4) Hypermultiplets*, DESY, Hannover and Maryland preprint, DESY-95-082, ITP-UH-15/95 and UMDEPP 95-116, April 1995; hep-th/9504077.
- [26] N. Dragon, Z. Phys. **C2** (1972) 29.
- [27] P. S. Howe, J. Phys. **A12** (1979) 393.
- [28] P. S. Howe and G. Papadopoulos, Class. and Quantum Grav. **4** (1987) 33.
- [29] S. V. Ketov and S.-O. Moch, Class. and Quantum Grav. **10** (1994) 11.
- [30] B. de Wit, R. Philippe and A. van Proeyen, Nucl. Phys. **B219** (1983) 143.
- [31] B. de Wit and A. van Proeyen, Nucl. Phys. **B245** (1984) 89.
- [32] S. Samuel, Nucl. Phys. **B245** (1984) 127.
- [33] A. van Proeyen, *Vector multiplets in $N=2$ supersymmetry and its associated moduli spaces*, Leuven preprint KUL-TF-95/39, November 1995; hep-th/9512139.
- [34] W. Siegel, Class. and Quantum Grav. **2** (1985) L41.
- [35] I. Antoniadis, C. Bachas and C. Kounnas, Phys. Lett. **242B** (1990) 185.
- [36] S. V. Ketov, *2d, $N=2$ and $N=4$ supergravity and the Liouville theory in superspace*, Hannover preprint ITP-UH-01/96, February 1996; hep-th/9602038; to appear in Phys. Lett. B (1996).
- [37] S. V. Ketov and I. V. Tyutin, JETP Lett. **39** (1984) 703.

- [38] B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. **B312** (1989) 58.
- [39] S. V. Ketov and S. J. Gates Jr., Phys. Rev. **D52** (1995) 2278.
- [40] E. A. Ivanov and A. A. Sutulin, Nucl. Phys. **B432** (1994) 246.
- [41] E. A. Ivanov, Nucl. Phys. B (Proc. Suppl.) **49** (1996) 350.
- [42] M. Pernici und P. van Nieuwenhuizen, Phys. Lett. **169B** (1986) 381.
- [43] K. Schoutens, Nucl. Phys. **B292** (1987) 150.
- [44] J. Wess, Acta Physica Austriaca, **41** (1975) 409.
- [45] S. J. Gates, Jr., Nucl. Phys. **B238** (1984) 349.